Ex.3.3 - Derivation of the Penman equation

We start by writing E and H using their aerodynamic formulations, and assuming that $r_{av} = r_{ah}$. In doing so, we also convert q to e, using the fact that :

$$q_a \simeq \frac{\epsilon e_a}{p_a} \tag{1}$$

For a saturated surface ($e_0 = e_s(T_0)$), this leads to

$$H = \rho_a \, c_p \, \frac{(T_0 - T_a)}{r_a} \tag{2}$$

$$E = \rho_a \, \frac{\epsilon}{p_a} \, \frac{(e_s(T_0) - e_a)}{r_a} \tag{3}$$

$$\tag{4}$$

E can be written as

$$E = \rho_a \, \frac{\epsilon}{p_a} \, \frac{(e_s(T_0) - e_s(T_a) + e_s(T_a) - e_a)}{r_a} \tag{5}$$

We introduce the drying power E_A :

$$E_A = \frac{\rho_a \, \epsilon}{p_a \, r_a} \, (e_s(T_a) - e_a) \tag{6}$$

Note that r_a depends on horizontal wind speed. It follows that :

$$E = \rho_a \, \frac{\epsilon}{p_a} \, \frac{(e_s(T_0) - e_s(T_a))}{r_a} + E_A \tag{7}$$

The next step is to linearize $e_s(T_0)$ using a first order limited development around T_a, which is close to T_0 :

$$e_s(T_0) \simeq e_s(T_a) + e'_s(T_a)\,(T_0 - T_a) = e_s(T_a) + \Delta\,(T_0 - T_a) \tag{8}$$

This leads to

$$E = \rho_a \, \frac{\epsilon}{p_a} \, \frac{\Delta\,(T_0 - T_a)}{r_a} + E_A \tag{9}$$

We then use the relation between $(T_0 - T_a)$ and H :

$$E = \frac{\epsilon\,\Delta}{c_p\,p_a}\,H + E_A \tag{10}$$

and the energy budget equation :

$$R_n = H + LE + G \tag{11}$$

The downward heat flux G into the soil (counted here positively if the flux cools the surface) exhibits a very strong diurnal cycle, with values that can exceed 100 W.m^{-2} at noon, and conversely very negative values (warming the surface) at night. This flux depends a lot on surface temperature, which we want to eliminate. It can be neglected if we work on average over one day, or any multiple of one day. **This is a very important validity condition for the Penman equation**. In such conditions, the surface energy budget can be simplified as :

$$R_n = H + LE \tag{12}$$

so that

$$E = \frac{\epsilon\,\Delta}{c_p\,p_a}\,(R_n - LE) + E_A \tag{13}$$

Introducing the psychrometric "constant" $\gamma = (c_p\, p_a)/(\epsilon L)$, we get

$$\frac{\epsilon\, \Delta}{c_p\, p_a} = \frac{\Delta}{\gamma\, L} \tag{14}$$

$$E = \frac{\Delta}{\gamma} \left(\frac{R_n}{L} - E \right) + E_A \tag{15}$$

Re-arranging this expression to isolate E gives the Penman equation :

$$E = \frac{\Delta}{\Delta + \gamma}\, \frac{R_n}{L} + \frac{\gamma}{\Delta + \gamma}\, E_A \tag{16}$$

Remark 1 : the above development assumes that R_n is independent from T_0, which is far from true, since it includes the upward long-wave radiation from the surface :

$$R_{lu} = \varepsilon_s \sigma T_0^4 \tag{17}$$

Most often, this term is estimated using T_a instead of T_0. In particular, it is the case in the FAO report on crop evaporation. A better approximation could be a achieved using the first-order limited development around T_a :

$$T_0^4 \simeq T_a^4 + 4T_a^3(T_0 - T_a) \tag{18}$$

Introducing the net short-wave radiation $R_{sn} = (1 - a_s)R_{sd}$, the energy budget equation can be rewritten as

$$R_{sn} + \varepsilon_s \sigma T_a^3 (T_0 - T_a) = \rho_a\, c_p\, \frac{(T_0 - T_a)}{r_a} + LE \tag{19}$$

From this, we can find $(T_0 - T_a)$ as a function of R_{sn}, T_a, and E, then proceed from Eq. 9.

Remark 2 : to derive the Penman-Monteith (1965) equation for unstressed vegetation, we follow the same sequence as for the Penman equation, but we use the following initial expression of E, depending on the minimum stomatal resistance r_0 :

$$E = \rho_a\, \frac{\epsilon}{p_a}\, \frac{(e_s(T_0) - e_a)}{r_a + r_0} \tag{20}$$

what leads to

$$E = \frac{\Delta\, \frac{R_n}{L} + \gamma\, E_A}{\Delta + \gamma \left(\frac{r_a + r_0}{r_a} \right)} \tag{21}$$

The FAO report from Allen et al. (1986) defines the reference ET, ET_0, from the Penman-Monteith equation, with $r_a = 208\ /\overline{u}(z = 2m)$ and $r_0 = 70$ s.m^{-1}

Actual ET can also be estimated by the the Penman-Monteith equation, by accounting the effects of environmental stresses and vegetation properties (albedo, physiology (r_0), LAI, height and roughness) owing to appropriate resistance formulations.