Vertical ALE coordinates and associated splitting error

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2D x-z tracer's advection

- In an incompressible fluid.

$$\begin{cases} \dfrac{\partial q}{\partial t} + u\dfrac{\partial q}{\partial x} + w\dfrac{\partial q}{\partial z} = 0 \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0 \end{cases}$$

In generalized vertical coordinates

$$z \to s(x,z,t), h = \frac{\partial z}{\partial s}, \Omega = \left.\frac{\partial s}{\partial t}\right|_z + u\left.\frac{\partial s}{\partial x}\right|_z + w\frac{\partial s}{\partial z}$$

$$\begin{cases} \dfrac{\partial hq}{\partial t} + \left.\dfrac{\partial huq}{\partial x}\right|_s + \dfrac{\partial \Omega q}{\partial s} = 0 \\ \dfrac{\partial h}{\partial t} + \left.\dfrac{\partial hu}{\partial x}\right|_s + \dfrac{\partial \Omega}{\partial s} = 0 \end{cases}$$

Generalized vertical coordinates

$$\frac{\partial h}{\partial t} + \left.\frac{\partial hu}{\partial x}\right|_s + \frac{\partial \Omega}{\partial s} = 0$$

- Quasi Eulerian: prescribe $h(x,s,t)$ or $\frac{\partial h}{\partial t}$ and deduce Ω
- Lagrangian: set $\Omega = 0$
- ALE: Keep $\frac{\partial h}{\partial t}$ close to $-\frac{\partial hu}{\partial x}$ and try to maintain h between minimum and maximum values, with sufficient regularity

Generalized vertical coordinates

In practice, h may become small:

- Use an unconditionnaly stable (implicit) vertical advection scheme:

$$\frac{(hq)^{n+1} - (hq)^n}{\Delta t} + \frac{\partial huq^n}{\partial x} + \frac{\partial \Omega q^{n+1}}{\partial s} = 0$$

→ Not accurate (too dissipative)

- Perform a directional splitting scheme:

$$\frac{(hq)^\star - (hq)^n}{\Delta t} + \frac{\partial huq^n}{\partial x} = 0,$$

$$\frac{(hq)^{n+1} - (hq)^\star}{\Delta t} + \frac{\partial \Omega q^\star}{\partial s} = 0$$

with an unconditonnaly stable and accurate (i.e. non implicit) vertical advection scheme.

↓

Flux-Form semi-lagrangian scheme advection scheme or (by the RTT), equivalent to remap $q^\star$ from $h^\star$ to h^{n+1} (since $\frac{h^{n+1} - h^\star}{\Delta t} + \frac{\partial \Omega}{\partial s} = 0$)

NB: using a (high-order) semi-lagrangian scheme without splitting is not stable

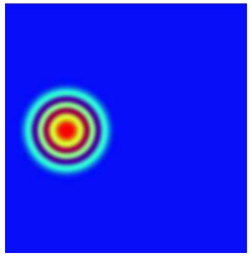
Back to a fixed vertical grid $h =$ Constant

Directional splitting:

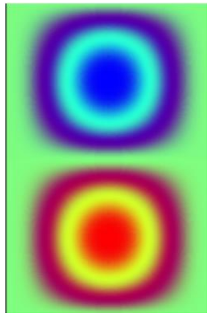
$$\frac{(q)^{\star} - (q)^{n}}{\Delta t} + \frac{\partial u q^{n}}{\partial x} = 0,$$

$$\frac{(q)^{n+1} - (q)^{\star}}{\Delta t} + \frac{\partial w q^{\star}}{\partial z} = 0 \text{ (using Lax-Wendroff)}$$

Leveque test case, $u(x, z, t) = \sin(\pi x)^2 \sin(2\pi z) \cos(\pi t / T)$



$q(t = 0)$



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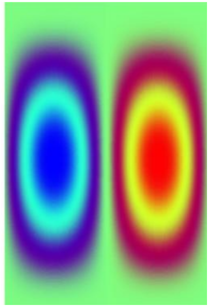
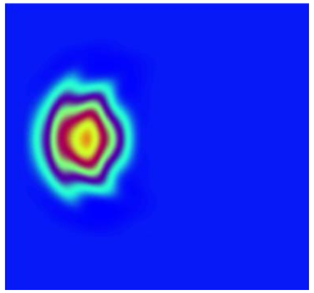


$w(t = 0)$



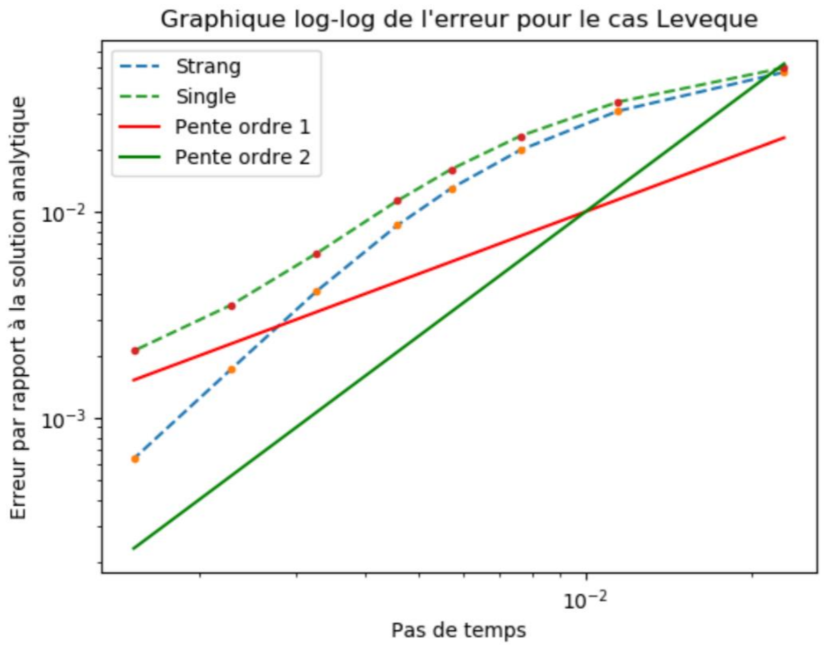
$q(t = 2T)$

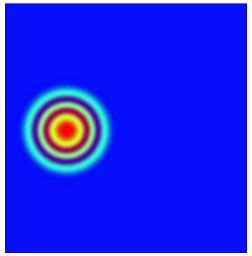
Back to a fixed vertical grid $h =$ Constant

From Lie to Strang directional splitting:

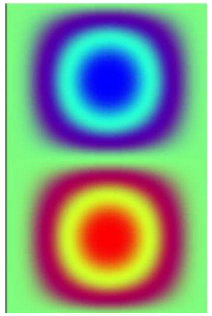
$$\frac{(q)^{\star} - (q)^{n}}{\Delta t/2} + \frac{\partial u q^{n}}{\partial x} = 0,$$
$$\frac{(q)^{\star\star} - (q)^{\star}}{\Delta t} + \frac{\partial w q^{\star}}{\partial z} = 0 \text{ (using Lax-Wendroff) },$$
$$\frac{(q)^{n+1} - (q)^{\star\star}}{\Delta t/2} + \frac{\partial u q^{\star\star}}{\partial x} = 0.$$

Leveque test case, $u(x,z,t) = \sin(\pi x)^2 \sin(2\pi z) \cos(\pi t/T)$





$q(t=0)$



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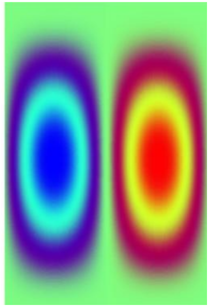

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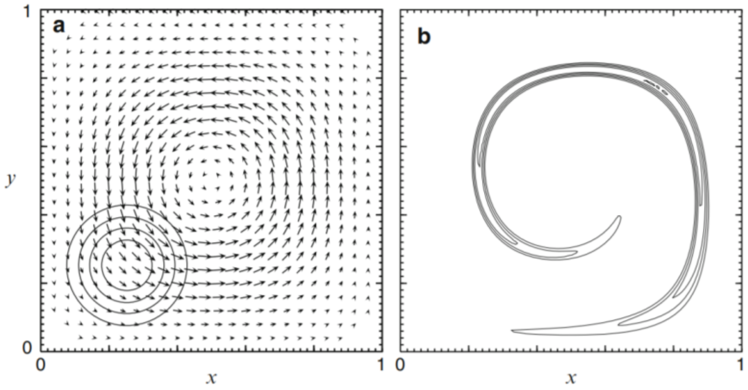

Back to a fixed vertical grid $h =$ Constant

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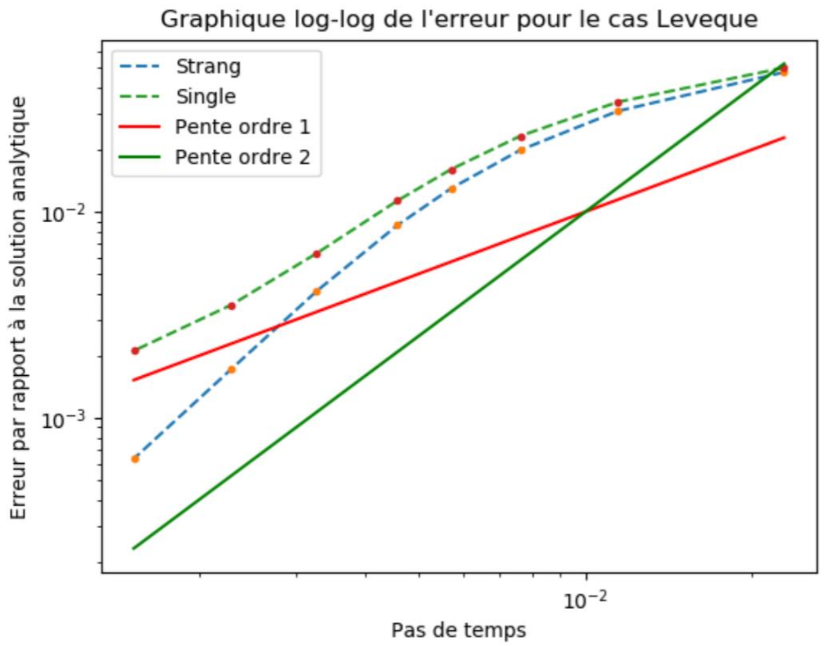
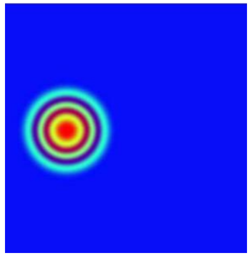
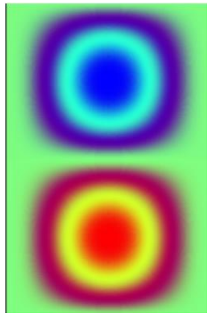
Leveque test case, $u(x,z,t) = \sin(\pi x)^2 \sin(2\pi z)\cos(\pi t/T)$

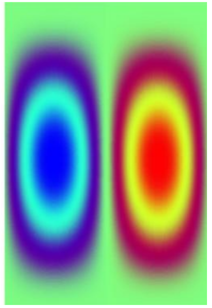




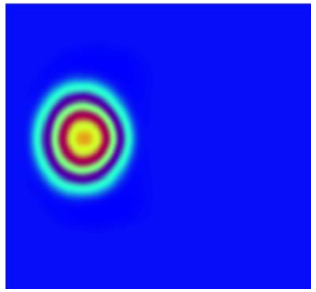
$q(t=0)$



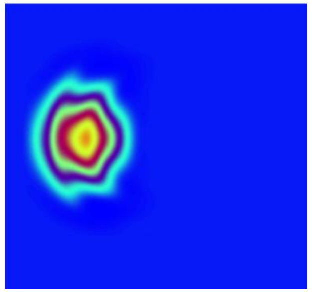
$u(t=0)$



$w(t=0)$



$q(t=2T)$



$q(t=2T)$

Back to a moving vertical grid ...

With a vertical Courant number exceeding 1 (at some points in space and time)

$$\frac{|\Omega|}{h}\Delta t > 1$$

$$\frac{(hq)^\star - (hq)^n}{\Delta t} + \frac{\partial huq^n}{\partial x} = 0, \qquad \text{RK3}$$

$$\frac{(hq)^{n+1} - (hq)^\star}{\Delta t} + \frac{\partial \Omega q^\star}{\partial s} = 0 \qquad \text{Vertical remapping}$$

Lie and Strang versions

Back to a moving vertical grid …

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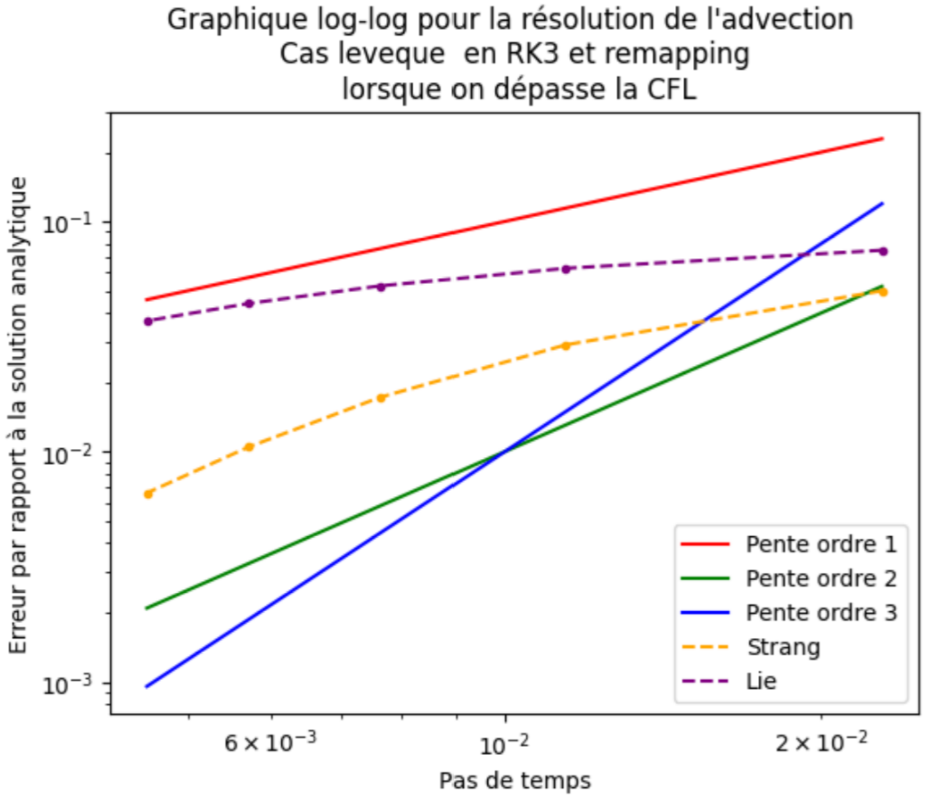
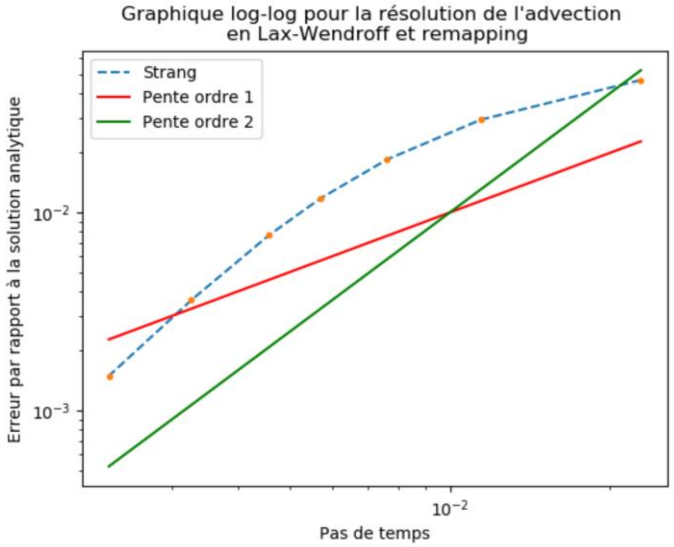
$$\frac{(hq)^{n+1} - (hq)^\star}{\Delta t} + \frac{\partial \Omega q^\star}{\partial s} = 0 \qquad \text{Vertical remapping}$$

Lie and Strang versions

Leveque test case





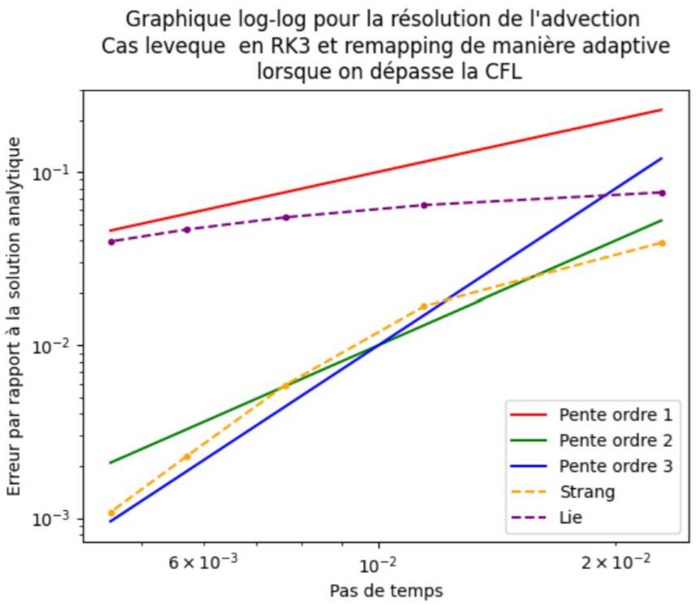
Same results for LW+remapping

A la Sasha ... $\Omega = \Omega_e + \Omega_i, \frac{|\Omega_e|}{h} < 1$

$$\frac{(hq)^\star - (hq)^n}{\Delta t} + \frac{\partial huq^n}{\partial x} + \frac{\partial \Omega_e q^n}{\partial s} = 0, \quad \text{RK3}$$

$$\frac{(hq)^{n+1} - (hq)^\star}{\Delta t} + \frac{\partial \Omega_i q^\star}{\partial s} = 0 \quad \text{Vertical remapping}$$

Leveque test case



Strang splitting + Adaptive vertical advection allows to recover third order accuracy

What’s next ?

- Perform more ocean oriented idealized advection test cases (with appropriate ALE choices)
- Tests in NEMO/CROCO are possible (the Adaptive option is already available)
- Use estimation of the splitting error (cross-derivatives terms) as an additional requirement for the design of ALE ?