1 Formulation of ALE

Omitting the velocity equation, the basic algorithm writes

$$\begin{array}{rcl} \frac{\partial h}{\partial t} & = & -\Delta t \left[\nabla_\sigma (hu) + \Delta_\sigma w^{(\dot{\sigma})}\right] \\ \frac{\partial hC}{\partial t} & = & -\Delta t \left[\nabla_\sigma (huC) + \Delta_\sigma (w^{(\dot{\sigma})} C)\right] \end{array} \quad (1)$$

and it can be implemented as it is. So the ALE methodology could be based on (1) and it seems to me that it is how it is done in NEMO or MPAS-O (i.e. *without directional splitting*). This leads to:

1	Prescribe $\Delta_\sigma w^{\text{grid}}$	
2	$[\Delta_\sigma w^{(\dot{\sigma})}]^{(n)} = -[\Delta_\sigma w^{\text{grid}}]^{(n)} - \nabla_\sigma [hu]^{(n)}$	diagnose dia-surface velocity
3	$h^{(n+1)} = h^{(n)} + \Delta t [\Delta_\sigma w^{\text{grid}}]^n$	update thickness (or $h^{(n+1)} = h^n - \Delta t \nabla_\sigma (hu)^n - \Delta t [\Delta_\sigma w^{(\dot{\sigma})}]^n$)
4	$h^{n+1} C^{n+1} = h^n C^n - \Delta t \nabla_\sigma (huC)^n - \Delta t \Delta_\sigma (w^{(\dot{\sigma})} C^n)$	Tracers' advection

The main problem with formulation (1) is that to be able to handle large vertical Courant number (small layer thicknesses / large vertical velocity), the vertical advection scheme has to be unconditionally stable and most of unconditionally stable *Eulerian* advection schemes won't have a good accuracy.

In order to recover accuracy, one has to rely on a semi-lagrangian advection scheme (or equivalently a remapping scheme) for the vertical derivatives. Semi-Lagrangian advection (or remapping) can be viewed as an extension of direct space time advection schemes to long time steps. However, as it is known, when implemented like this (i.e. without splitting), only low order (i.e. first order) direct space time advection scheme would lead to a stable multidimensional scheme.

So, one has to rely on a directional splitting version of (1).

A general version of the horizontal - vertical splitting is:

1	$h^\dagger = h^n - \Delta t \nabla_\sigma (hu)^n$	Horizontal advection following Easter (1993)
2	$h^\dagger C^\dagger = h^n C^n - \Delta t \nabla_\sigma (huC)^n$	to preserve constancy
3	Prescribe $\Delta_\sigma w^{\text{grid}}$	
4	$[\Delta_\sigma w^{(\dot{\sigma})}]^{(n)} = -[\Delta_\sigma w^{\text{grid}}]^{(n)} - \nabla_\sigma [hu]^{(n)}$	diagnose dia-surface velocity
5	$h^{(n+1)} = h^{(n)} + \Delta t [\Delta_\sigma w^{\text{grid}}]^n$	update thickness (or $h^{(n+1)} = h^\dagger - \Delta t [\Delta_\sigma w^{(\dot{\sigma})}]^n$)
6	$h^{n+1} C^{n+1} = h^\dagger C^\dagger - \Delta t \Delta_\sigma (w^{(\dot{\sigma})} C^\dagger)$	Vertical advection (or remapping)

Quasi-Eulerian methods correspond to $[\Delta_\sigma w^{\text{grid}}] \propto \partial \eta / \partial t$ while ALE methods to $[\Delta_\sigma w^{\text{grid}}] = (h^{\text{target}} - h^n)/\Delta t$.

For performing (6) one has the choice between remapping schemes (a.k.a. Cell Integrated Semi Lagrangian (CISL) schemes in the atmospheric modeling community [5]) or (Flux Form) semi lagrangian schemes [6].

The above splitting is however only first order (in time) accurate. It should be able to recover second order accuracy by alternating the horizontal/vertical advection steps from one time step to another (i.e. ≈ Strang splitting, [6]). Another idea could be to use an approach like the one developed in [2] and to apply the remapping scheme, i.e. to use the directional splitting, only for the part of the "vertical velocity" $w^{(\dot{\sigma})}$ which exceeds the CFL of the time integration scheme.

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