Balance of the vertically averaged vorticity in OPA9

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Contents

1 Introduction

For various applications it is interesting to consider the vorticity equation. We describe here the on-line calculation of the various terms of the vorticity equation in OPA9. Those calculations have been introduced as an option in the code (using a CPP key "key_trd_vor") by Laurent Brunier during his stay at LPO (january-may 2004).

We considered only the balance for the vertically averaged vorticity. It should be easy for any user to integrate vertically instead of averaging, or to consider the average/integral over a layer of fluid. We chose to consider the vertically averaged vorticity

$$\overline{\zeta} = \frac{1}{H(x,y)} \int_{-H(x,y)}^{\eta} \zeta \, dz$$

rather than the vertically integrated one, because we wanted to study the so-called "JEBAR" term (Joint Effect of Baroclinicity And Relief) which appears only in the equation for the vertical average $\overline{\zeta}$. Looking back, it seems that perhaps the vertically integrated equation is easier to interpret. For instance, the advection of planetary vorticity "βV" does not appear clearly in the equation for $\overline{\zeta}$, so the exact calculation of this term has been added.

The vorticity diagnostics have been tested using the standard "EEL5" configuration (this is described in the present reference manual) and in realistic configurations (Brunier, METEO-FRANCE report, 2004).

2 Model equations

2.1 Notations

In this study, we will use these notations (vectors are in bold type) :

- (**i**,**j**,**k**) the basis, with **k** orthogonal to the geopotential surfaces,
- a the Earth radius,
- **U** = (u,v,w) velocity vestor, $\mathbf{U_h}$ = (u,v),
- P the pressure, P_s the surface pressure, P_b the bottom pressure and P_h the hydrostatic pressure,
- ρ the density,
- f the Coriolis parameter,
- $\mathbf{D^u} = (D^u_x, D^u_y)$ diffusion terms,
- ζ the two-dimensions relative vorticity,
- H(x,y) the topography,
- η the sea surface height (SSH),
- $\overrightarrow{rot}$ the horizontal curl $(A = (a_1, a_2))$: $\overrightarrow{rot}(\mathbf{A}) = \dfrac{\partial a_2}{\partial x} - \dfrac{\partial a_1}{\partial y}$,
- $\mathcal{J}(A,B)$ the Jacobian operator : $\mathcal{J}(A,B) = \dfrac{\partial A}{\partial x}\dfrac{\partial B}{\partial y} - \dfrac{\partial A}{\partial y}\dfrac{\partial B}{\partial x}$

This relation will be used :

$$\frac{\partial}{\partial \alpha}\int_a^b f d\beta = \int_a^b \frac{\partial f}{\partial \alpha} d\beta + \frac{\partial b}{\partial \alpha} f(b) - \frac{\partial a}{\partial \alpha} f(a)$$

2.2 Equations

The primitive equations used in OPA imply the following hypotheses :

- Spherical earth approximation

- Thin-shall approximation
- Turbulent closure hypothesis
- Boussinesq hypothesis
- Hydrostatic hypothesis
- Incompressibility hypothesis

2.2.1 Dynamical equations

The momentum equations used are :

$$\left\{\begin{array}{l}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=fv-\frac{1}{\rho_0}\frac{\partial P}{\partial x}+D_x^u\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-fu-\frac{1}{\rho_0}\frac{\partial P}{\partial y}+D_y^u\end{array}\right. \tag{1}$$

These equations can be written (cf appendix) :

$$\boxed{\left\{\begin{array}{l}\frac{\partial u}{\partial t}=(f+\zeta)v-w\frac{\partial u}{\partial z}-\frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2)-\frac{1}{\rho_0}\frac{\partial P_h}{\partial x}-g\frac{\partial \eta}{\partial x}+D_x^u\\ \frac{\partial v}{\partial t}=-(f+\zeta)u-w\frac{\partial v}{\partial z}-\frac{1}{2}\frac{\partial}{\partial y}(u^2+v^2)-\frac{1}{\rho_0}\frac{\partial P_h}{\partial y}-g\frac{\partial \eta}{\partial y}+D_y^u\end{array}\right.} \tag{2}$$

The right hand side is made of 8 terms :
- $-f\mathbf{k}\wedge\mathbf{U_h}$ Coriolis term,
- $-\zeta\mathbf{k}\wedge\mathbf{U_h}$ relative vorticity advection,
- $-\frac{1}{2}\nabla_h\mathbf{U_h}^2$ kinetic energy advection,
- $-w\frac{\partial\mathbf{U_h}}{\partial z}$ vertical advection,
- $-\frac{1}{\rho_0}\nabla_h P_h$ horizontal gradient oh hydrostatic pressure,
- $-g\nabla_h\eta$ horizontal gradient of SSH,
- $\mathbf{D^u}$ diffusions.

$\mathbf{D^u}$ corresponds to the diffusion terms which are made of :
- Vertical diffusion : $\mathbf{D^u}_1=\frac{\partial}{\partial z}\left(K_1\frac{\partial\mathbf{U_h}}{\partial z}\right)$ with K_1 a vertical eddy diffusivity factor
- lateral diffusion : $\mathbf{D^u}_2=K_2\nabla^2\mathbf{U_h}$ or $\mathbf{D^u}_2=K_2\nabla^4\mathbf{U_h}$ if we are in laplacian or bi-laplacian.

2.2.2 Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{3}$$

2.2.3 Boundary conditions

- Surface (z = η) :

$$w=\frac{\partial\eta}{\partial t}+\mathbf{U_h}|_{z=\eta}\cdot\nabla_h(\eta) \tag{4}$$

- Bottom (z = -H) :

$$w=-\mathbf{U_h}.\nabla_h(H) \tag{5}$$

Number	Name in netcdf file	Momentum equation	Vorticity equation
1	sovortPh	Hydrostatic pressure	JEBAR
2	sovortEk	Advection (kinetic energy)	Advection (kinetic energy)
3	sovozeta	Advection (ζ)	Advection (ζ)
4	sovortif	Coriolis term	Divergence
5	sovodifl	Lateral diffusion	Lateral diffusion
6	sovoadvv	Vertical advection	Vertical advection
7	sovodifv	Vertical diffusion	=0
8	sovortPs	Surface pressure	=0
9	sovortbv	Coriolis term	$\beta.V$ (integrated form)
10	sovowind	Wind stress	Wind stress
11	sovobfri	Bottom friction	Bottom friction
12	1st_mbre	$\frac{\partial\zeta}{\partial t}$	$\frac{\partial\zeta}{\partial t}$
13	sovorgap		difference between lhs and rhs

TAB. 1 – List of the terms calculated and written by the trdvor routine.

2.3 Vorticity equation

To construct vorticity equation, each term must be averaged over the depth ($\frac{1}{H(x,y)}\int_{-H(x,y)}^{\eta} ...dz$), then, we must take the curl of the momentum equations, that is to say $\frac{\partial V}{\partial x\partial t} - \frac{\partial U}{\partial y\partial t}$ with U, V the u and v averaged.

The vorticity equation can be written as : $\frac{\partial\zeta}{\partial t}$ = divergence (come from Coriolis term) + horizontal advection (relative vorticity + kinetic energy) + vertical advection + JEBAR (come from hydrostatic pressure term) + SSH + vertical diffusion + horizontal diffusion + bottom friction + wind stress

3 Implementation

To use the vorticity diagnostics, the user has to compile the code with the "key_trd_vor". When running, the three-dimensional trends of the momentum equation are saved in arrays utrd and vtrd (the same happens when the key_trd_dyn is active). This means that this option uses a significant amount of memory. At each time step the vertical average and the curl are taken in routine trd_vor (trdvor.F90 module) and accumulated in a netcdf file. The terms are averaged in time over "ntrd" time steps (namelist parameter), and tendencies are estimated over the same period of time.

The ouput file name has "vort" in its name (EEL5-02_1d_010101_010105_vort.nc for instance). It contains the different terms listed in Table 1. Because of a limitation of the ioipsl package, the tendency (term 12) and the misfit (term 13) have been divided by ntrd before writing in the file (as if they were time-averages). To get the true value, the user must multiply them by ntrd again. The module added is trdvor.F90 (which include trdvor_ncinit.h90 and trdvor_ncwrite.h90).

4 Validation in an idealized case

The configuration used to validate the diagnostics is derived from J. Verron's thesis [4]. The CPP keys used are : key_eel_r5, key_dynspg_fsc, key_zdfcst, key_obc, key_trd_vor.

4.1 Characteristics of the configuration

Channel dimensions are 202×104 (Ox,Oy) grid points with 5 km resolution. The underwater seamount is centered at the third of the channel length and half the channel width. It is represented by the gaussian $h = h_m \exp^{(r/R_0)^2}$, with a radius R_0=50 km. In this study, the bump height is 240m (h_m=240m). The channel depth is 4000 m. The channel has 40 vertival levels with a vertical grid spacing larger near the surface than near the bottom. The latitude is around 35°N (northern hemisphere).

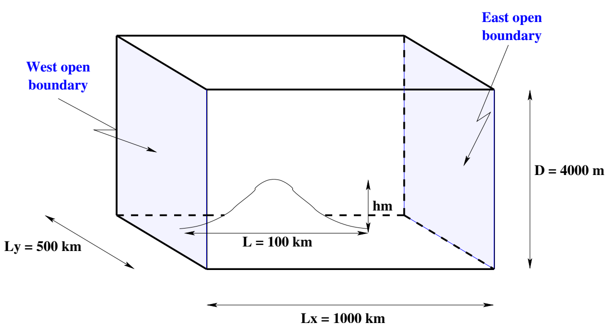


Figure 1 – Channel

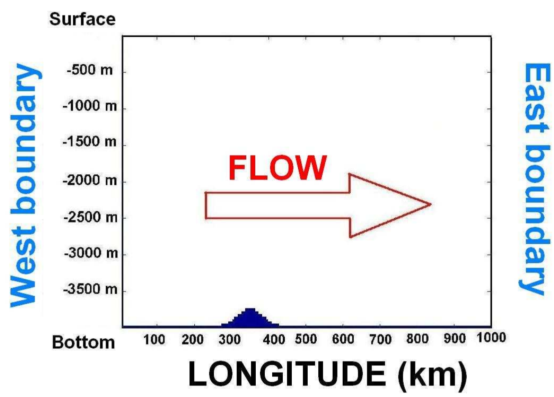


Figure 2 – Zonal section

This study uses open boundaries. The values at the open boundaries are fixed (those for the initial condition). The barotropic velocity is uniform eastward (0.1 $m.s^{-1}$). The baroclinic velocity normal to the boundaries is zero. In the first experiments, the temperature and salinities are uniform (T=10°C, S=35.5 PSU). The free surface is initialized to ensure geostrophic balance of the initial velocity field.

Where the flow feels the seamount, two eddies are generated. The first is cyclonic (positive vorticity) on the bump, the second anticyclonic (negative) in the lee of the bump. The cyclonic eddy is trapped on the bump (Taylor column if $\mu = \frac{h}{\varepsilon D} > \mu_0 (= O(1))$ where ε is the Rossby number ; in our study $\mu = 6$, and the Taylor column exists). The anticyclonic eddy can be advected to the East boundary [4].

4.2 Model solution

The next graphics shows the vorticity during 20 days of simulation, on a f-plane, using full steps topography. Negative vorticity (anticyclonic) is red, and positive (cyclonic) is blue. Isolines are separated by $5.10^{-7}s^{-1}$, from 5.10^{-7} to $35.10^{-7}s^{-1}$.

At first, two eddies are created and later the cyclonic eddy is advected. As we can see, the results are noisy, particularly on the bump.

4.3 Vorticity balance

The aim of this section is to verify the programs added to the model OPA9 for the vorticity budget (trdvor.F90). Here, we don't use stratification (T=10^{o}C), bottom friction nor wind stress.

Results are given after 2 days of simulation (ie when the eddy is well formed). For clarity of the graphics, the zero-contours are not drawn. Note that the plots show only the region around the

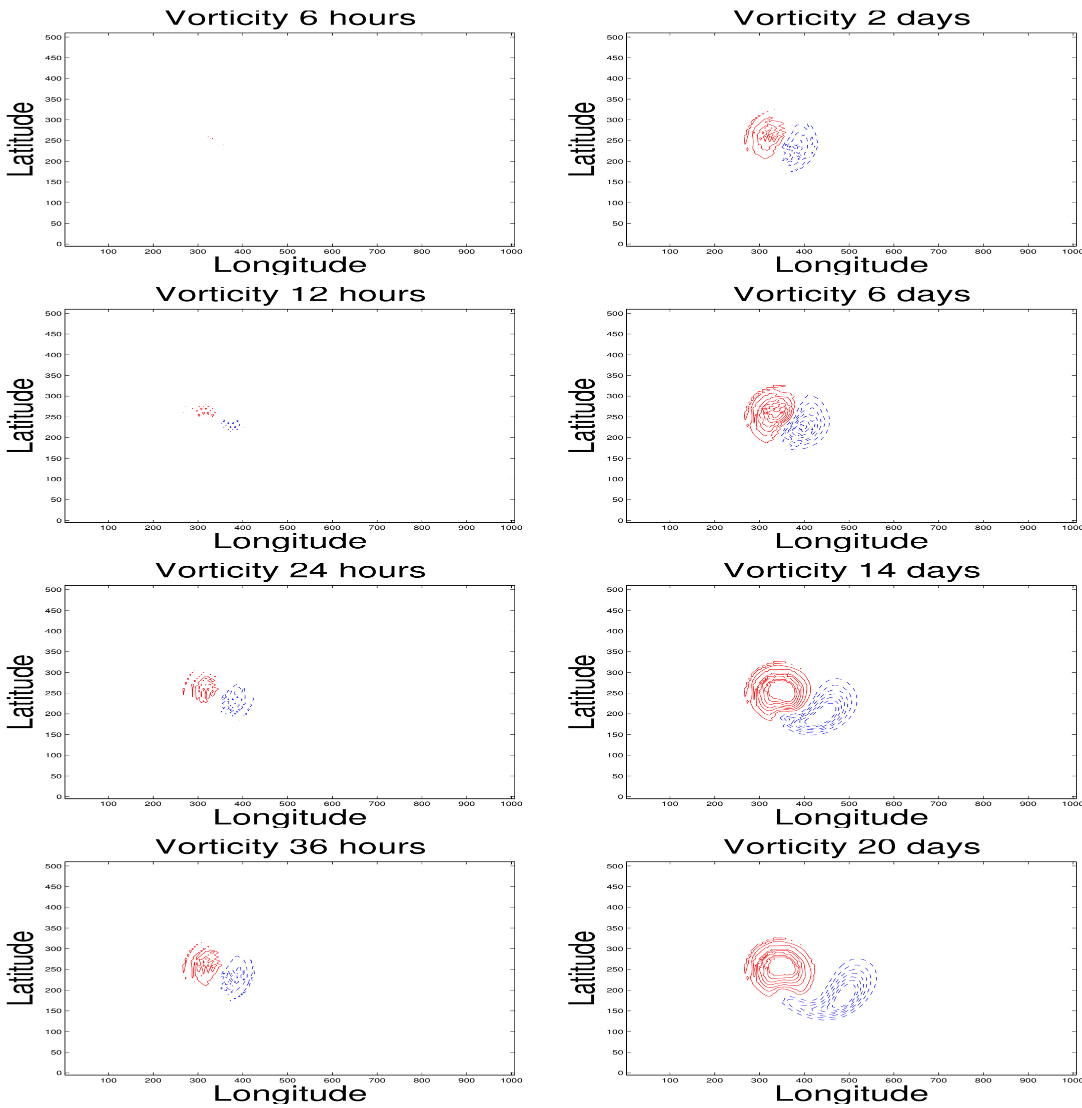


Figure 3 – Vertically averaged relative vorticity

bump. We have tried to make the contour interval identical for all plots to compare them (-1.10^{-11} to $+1.10^{-11}$ s^{-2}), excepted in particular cases.

For scaling, we use the following values. Note that at first order (in the initial conditions), density and velocities are uniform over the domain so that the scales when spatial derivatives are involved are those of perturbations (smaller).

- $\Delta X = \Delta Y = 5.10^3 m$; $\Delta Z = 10^2 m$
- $L = 5.10^5 m$
- $K_2 = 10^{10} m^2.s^{-1}$ (horizontal eddy diffusivity factor)
- $K_1 = 10^{-5} s^{-1}$ (vertical eddy diffusivity factor)
- $C_D = 4.10^{-3} m.s^{-1}$ (bottom drag coefficient factor)
- $H = 4.10^3 m$
- $f = 10^{-4} s^{-2}$
- $\Delta U = \Delta V = 10^{-3} m.s^{-1}$ for $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$
- $\Delta U = \Delta V = 10^{-2} m.s^{-1}$ for bottom $\frac{\partial}{\partial z}$
- $U = 10^{-1} m.s^{-1}$
- $W = 10^{-3}$; $\Delta W = 10^{-4}$
- $\rho_0 = 10^3$ kg.m^{-3} ; $\Delta\rho = \rho = 10^{-6}$

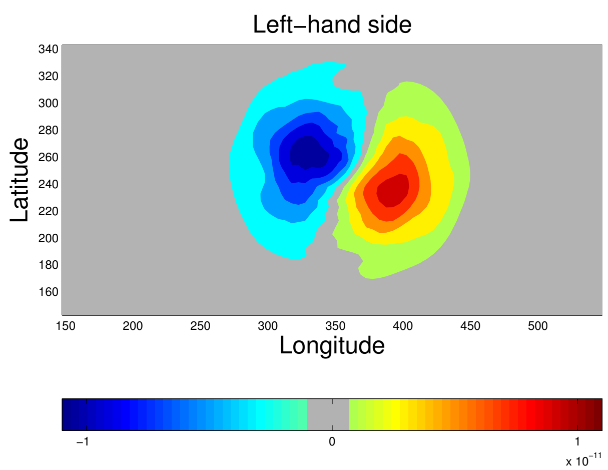
Scalings are calculated from the momentum equation (8). Then, they are averaged on the depth, and the curl is taken.

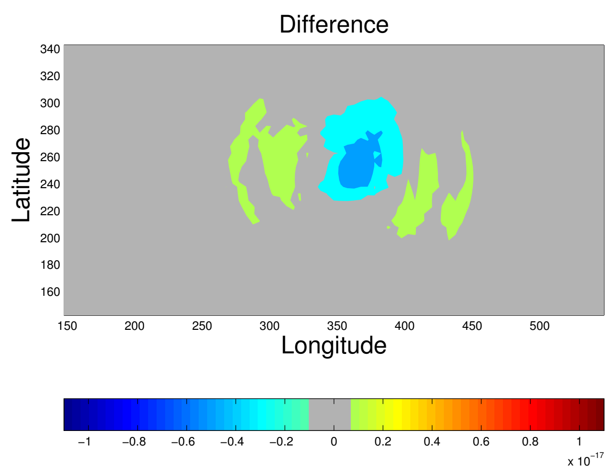
The vorticity equation can be written as : $\frac{\partial \zeta}{\partial t}$ = divergence + horizontal advection (relative vorticity + kinetic energy) + vertical advection + JEBAR + SSH + vertical diffusion + horizontal diffusion + bottom friction + wind stress

The left-hand side is $\frac{\partial \zeta}{\partial t}$ (left term), and the right-hand side is the sum of the other terms.

4.3.1 Equality of the lhs and rhs

The lhs was directly calculated from velocity, that is to say independently of the second. The first point is to verify the equality of the two sides of the equation. The figures show the lhs and the difference (lhs-rhs). The scale of the difference is 10^{-17}, six orders of magnitude smaller than either term, so the calculation is consistent.





4.3.2 Divergence (comes from the Coriolis term)

In this paragraph and the following ones, we estimate a scaling for each term of the rhs of the equation and compare it to the values calculated by the program.

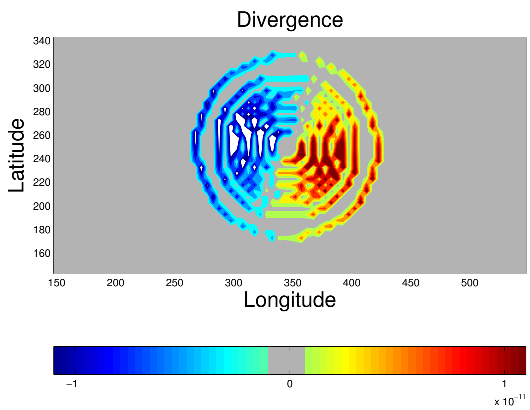


Figure 4 – Divergence

Scaling :

$$\mathcal{A} = -f\mathbf{k} \wedge \mathbf{U_h}$$

The vertical average doesn't change the scale of the term. The curl gives :

$$\mathcal{A}_{final} = f\frac{\Delta U}{\Delta X} = 10^{-4}.\frac{10^{-3}}{5.10^{3}} = 10^{-11}$$

$$O(\mathcal{A}_{final}) = 10^{-11}\mathrm{s}^{-2}$$

Moreover, we can use the expression found : $f\mathbf{U_h} \cdot \nabla_h(H)$, with $\beta = 0$ to determine the sign of this term. Before the bump, H decreases and U is Eastward. Consequently, this term in negative. And positive after, which corresponds with the plot. Numerical noise is very present. This will be adressed later.

4.3.3 Relative vorticity advection

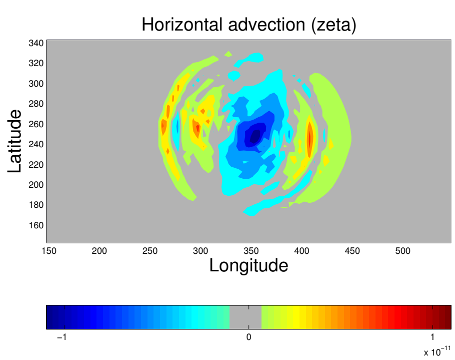


Figure 5 – Relative vorticity

Scaling :

$$\mathcal{B} = -\zeta\mathbf{k} \wedge \mathbf{U_h} = \frac{\Delta U}{\Delta X}.U$$

The vertical average doesn't change the scale of the term. The curl gives :

$$\mathcal{B}_{final} = \frac{\Delta U}{\Delta X \Delta X}.U + \left(\frac{\Delta U}{\Delta X}\right)^2 = \frac{10^{-3}}{(5.10^{3})^2}.10^{-1} + \left(\frac{10^{-3}}{(5.10^{3})^2}\right)^2$$

$$O(\mathcal{B}_{final}) = 10^{-12} + 10^{-14} = 10^{-12}\mathrm{s}^{-2}$$

4.3.4 Kinetic energy advection

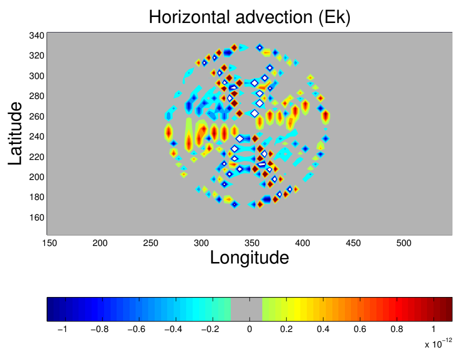


Figure 6 – Kinetic energy

Caution ! Axis values changed : they stretch from -1.10^{-12} to 1.10^{-12}, and only for this plot.
Scaling :

$$\mathcal{C} = \frac{1}{2}\nabla_h \mathbf{U_h}^2 = \frac{U.\Delta U}{\Delta X}$$

It is the same as relative vorticity advection.

$$O(\mathcal{C}_{final}) = 10^{-12}\text{s}^{-2}$$

On the plot, we can see that this term is a little bit smaller than the others.

4.3.5 Vertical advection

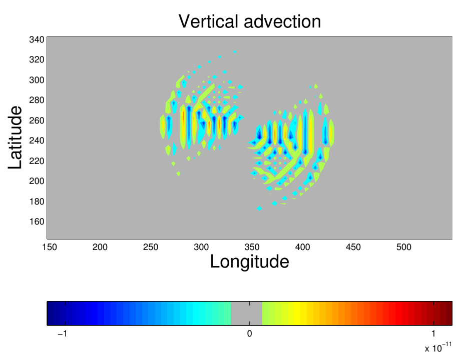


Figure 7 – Vertical advection

Scaling :

$$\mathcal{D} = -w\frac{\partial \mathbf{U_h}}{\partial z} = W.\frac{\Delta U}{\Delta Z}$$

and

$$\mathcal{D}_{final} = \frac{\Delta W}{\Delta X}\frac{\Delta U}{\Delta Z} + W.\frac{\Delta U}{\Delta Z \Delta X} = \frac{10^{-4}}{5.10^3}.\frac{10^{-3}}{10^2} + 10^{-3}.\frac{10^{-3}}{10^2.5.10^3}$$

$$O(\mathcal{D}_{final}) = 10^{-13} + 10^{-12} = 10^{-12}\text{s}^{-2}$$

4.3.6 Horizontal diffusion

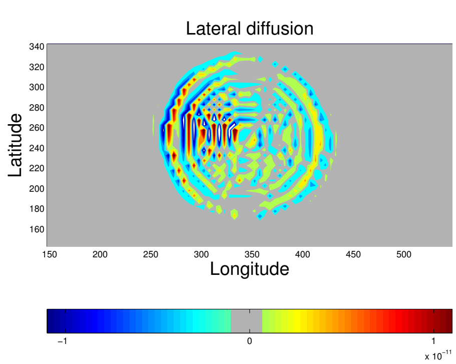


Figure 8 – Horizontal diffusion

Scaling :

$$\mathcal{H} = K_2 \nabla^4(\mathbf{U_h})$$

The curl gives :

$$\begin{aligned}
\mathcal{H}_{final} &= K_2\frac{\Delta U}{(\Delta X)^4.\Delta X} + \frac{\Delta H}{H^2 \Delta X} K_2 \nabla^4(\mathbf{U_h}) \\
&= 10^{10}\frac{10^{-3}}{(5.10^3)^5} + \frac{10}{16.10^6 5.10^3}.10^{10}\frac{10^{-3}}{(5.10^3)^4} \\
&= 10^{12} + 10^{-18}
\end{aligned}$$

$$O(\mathcal{H}_{final}) = 10^{-12}\text{s}^{-2}$$

4.3.7 Other terms

- **JEBAR** This term comes from hydrostatic pressure term. Without stratification, density is constant, and consequently this term doesn't depend on x or y. The gradient is zero, and this term doesn't play a part in the vorticity equation.
- **SSH** This term is independant of the vertical. Consequently, it is zero in the equation.
- **Vertical diffusion** Taking the vertical average, we show that this term is only the sum of the wind stress and the bottom friction. Those are calculated separately, so that the remaining "vertical diffusion" in the vorticity diagnostics file is identically zero.

4.3.8 Wind stress

To obtain a non-zero wind forcing, we use a sinusoïdal stress $\sin(\frac{\pi y}{L})$ with L the width of the channel (=500km). In the model, this term is calculated with its exact formulation $\frac{\tau}{\rho_0 \Delta z_1}$, Δz_1 is the thickness of the first level, near the surface.

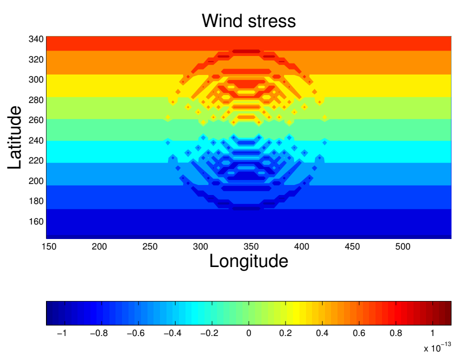


Figure 9 – Wind stress

Caution ! values changed : from -1.10^{-13} to 1.10^{-13} s^{-2}.

Scaling : The term is calculated as :

$$I = \frac{\tau_0}{\rho_0 \Delta z_1} \sin(\frac{\pi y}{L})$$

The average divides the term by H, and the curl gives :

$$\begin{aligned} I_{final} &= \frac{\Delta H}{H^2 \Delta X} \frac{\tau_0}{\rho_0} \sin(\frac{\pi y}{L}) + \frac{1}{H} \frac{\tau_0 \pi}{\rho_0 L} \cos(\frac{\pi y}{L}) \\ &= \frac{10}{16.10^6 5.10^3} \frac{10^{-1}}{10^3} + \frac{1}{4.10^3} \frac{10^{-1}.3.14}{10^3.5.10^5} \end{aligned}$$

and

$$I = 10^{-14} + 10^{-13} = 10^{-13} s^{-2}$$

The value $\Delta U = 4.10^{-3}$ was found from the model.

4.3.9 Bottom friction

Like for wind stress, we use a particular formulation for this term : $\frac{C_D.U}{\Delta z_{fond}}$ with Δz_{fond} the thickness of the last level before the bottom. We have used linear friction (namelist option).

Scaling : In the model, this term is calculated using the expression :

$$\mathcal{J} = \frac{C_D.U}{\Delta z_{fond}}$$

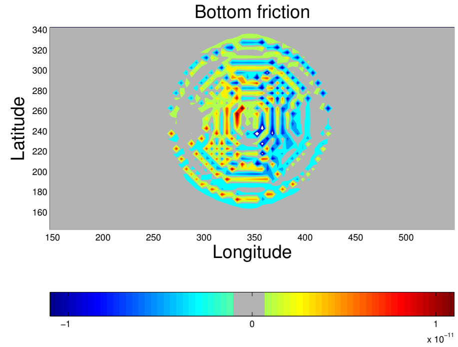


Figure 10 – Bottom friction

The average :

$$\mathcal{J}_{moy} = \frac{1}{H} C_D.U$$

And with the curl, we have :

$$\begin{aligned}
\mathcal{J}_{final} &= \frac{\Delta H}{H^2 \Delta X} C_D.U + \frac{1}{H} C_D \frac{\Delta U}{\Delta X} \\
&= \frac{10}{16.10^6 5.10^3} 4.10^{-3}.10^{-1} + \frac{1}{4.10^3} 4.10^{-3} \frac{10^{-3}}{5.10^3} \\
&= 10^{-13} + 10^{-13}
\end{aligned}$$

$$O(\mathcal{J}_{final}) = 10^{-13} \mathrm{s}^{-2}$$

At the bottom, the thickness of the last level is about thirty meters, and the scaling of $\Delta U = 10^{-2}$ m.s^{-2} has been found with the model.

4.3.10 Stratification

Without stratification, hydrostatic pressure term (JEBAR) is zero. To test this term, a linear stratification for the temperature has been added.

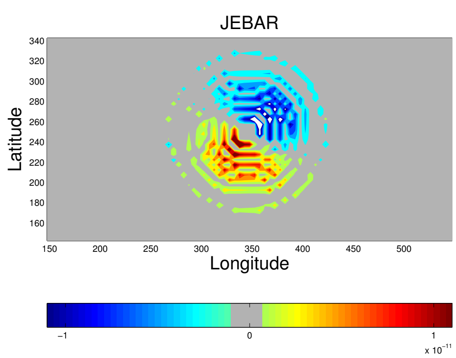


Figure 11 – JEBAR

Scaling :

$$\mathcal{E} = -\frac{1}{\rho_0} \nabla_h P_h \text{ with } P_h = \int_{-H}^{\eta} \rho g dz$$

The vertical average doesn't change the scaling, and the curl gives :

$$\begin{aligned}
\mathcal{E}_{final} &= -\frac{\Delta}{\Delta x} \left(\frac{1}{\rho_0} \nabla_h \int_{-H}^{\eta} \rho g dz \right) \\
&= \frac{1}{\rho_0} \frac{\Delta \rho}{\Delta x \Delta x} g.H = \frac{1}{10^3} \frac{10^{-6}}{25.10^6} 10.4.10^3
\end{aligned}$$

$$O(\mathcal{E}_{final}) = 10^{-12} \mathrm{s}^{-2}$$

4.3.11 Main balance of the equation

The configuration is a stratified configuration with friction. Contour intervals are equal, to compare the plots.

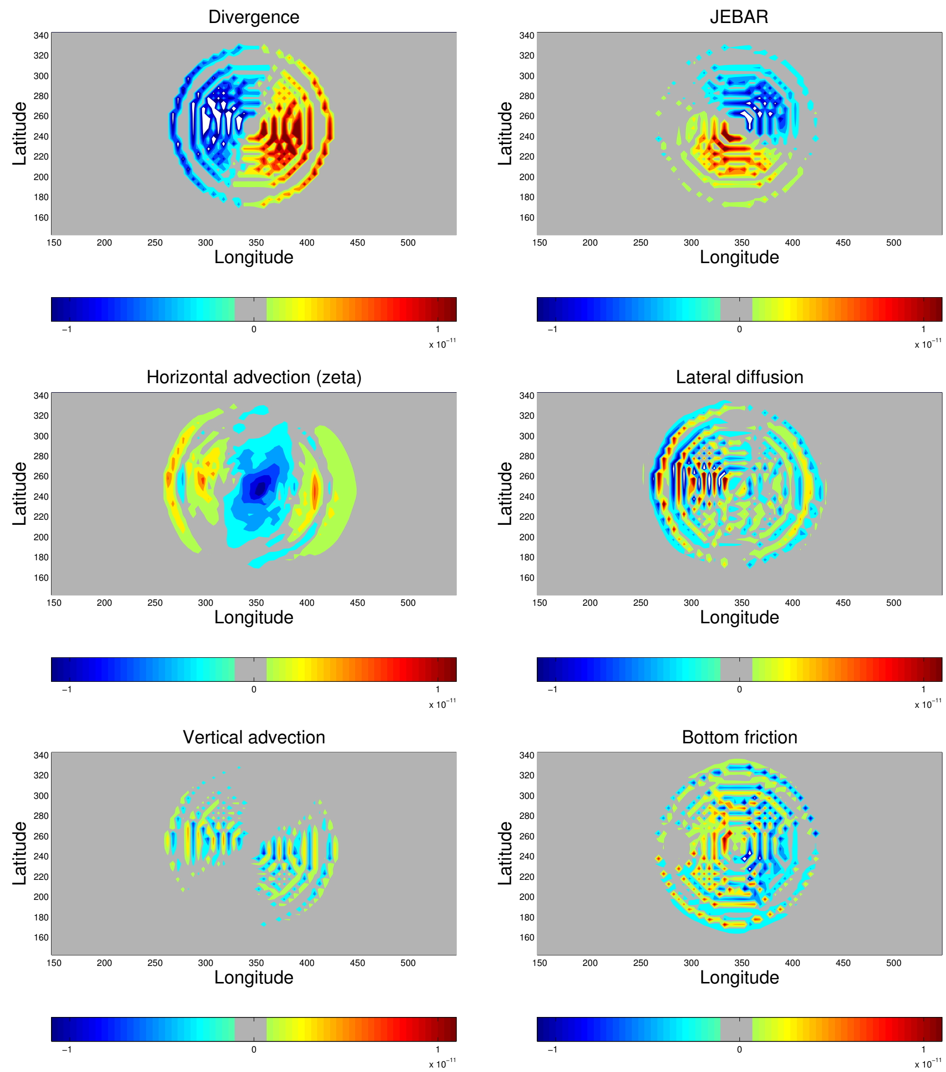


Figure 12 – Main terms

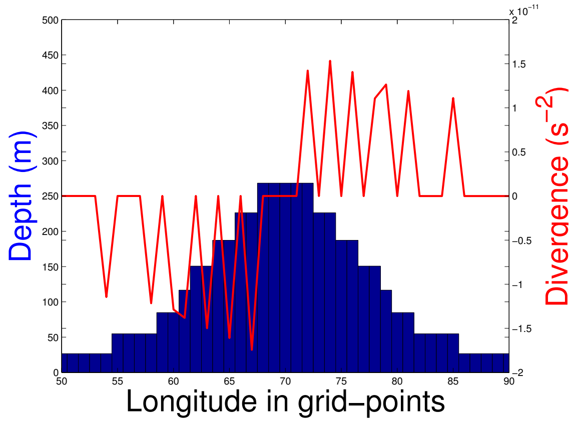
We can trust these results, even if they are noisy. The main criterion is the equality of the two sides of the equation, because they were calculated separately. The next sections deals with numerical noise.

4.4 Numerical noise

4.4.1 Effect of the full step topography

During the eddy formation, we saw that the noise happened on the bump. The previous experiments were performed with full steps topography, where the bathymetry is represented as a series of steps. Changing vertical coordinates, we can improve the smoothness of the solution. We will compare full steps and partial steps, first for the divergence term, and then for others.

We have superimposed the topography (blue) and a section of the divergence term, in the middle of the channel.



The red curve shows the noise. It is interesting to note it appears when topography changes, when there is a “jump". It is understandable if we take the notation of this term found in appendix ($f\mathbf{U_h} \cdot \nabla_h(H)$). Indeed, this term in non-zero when H varies, which is not always the case with these coordinates. Each time the topography doesn't change between two neighbouring grid points, the term becomes zero, and the noise appears.

Figure 13 – Section and topography

4.4.2 Using partial steps

To try to decrease the noise, we use partial steps. Let's consider the same plot as the previous one.

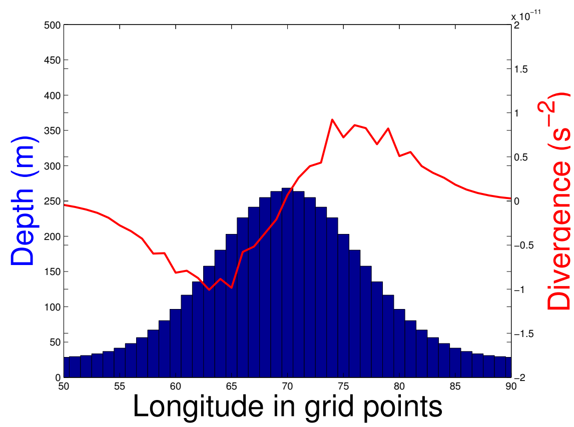


Figure 14 – Divergence term section and topography in partial steps

In this case, topography changes at every grid-point. Consequently, the noise decreases. this is confirmed by a comparison of the divergence term in full steps and partial steps :

4.4.3 Vorticity balance in partial steps

Let us now consider the main terms of the vorticity equation, in partial steps :

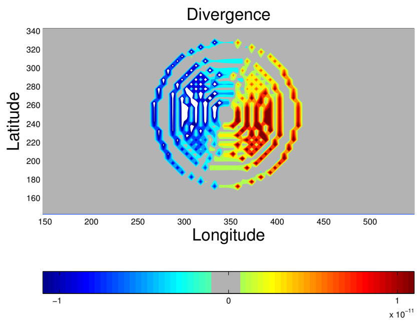


Figure 15 – Divergence term in full steps

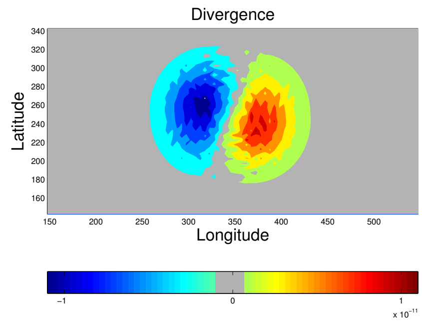


Figure 16 – Divergence term in partial steps

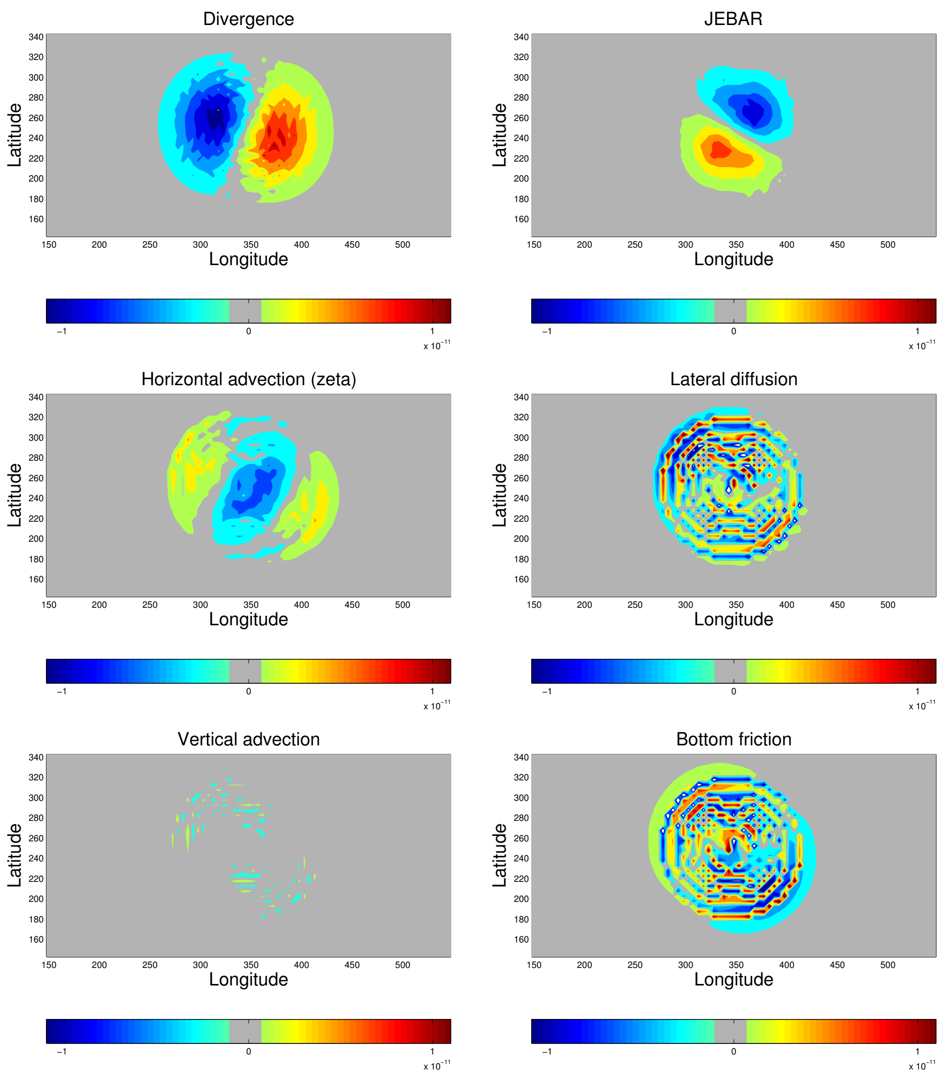


Figure 17 – Main terms in partial steps

These plots can be compared with those in full steps. They are similar for the values, but some, like JEBAR, are less noisy. Even if the improvement of some terms like vertical advection is not as important as for the divergence term, we can see with the next graphics that it is not negligeable :

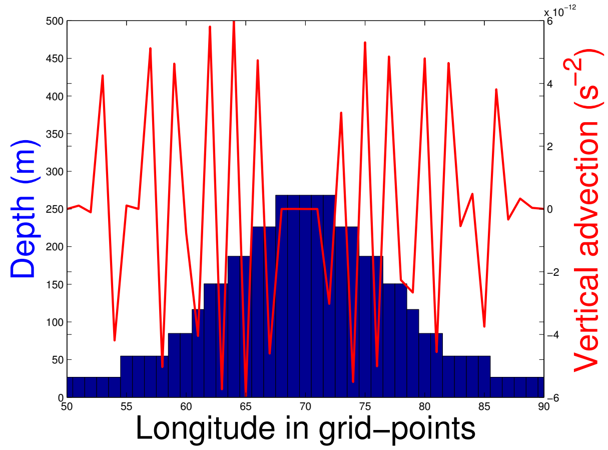


Figure 18 – Vertical advection

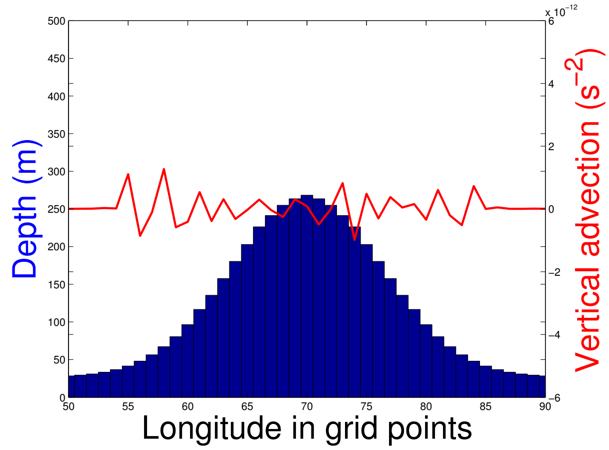


Figure 19 – Vertical advection in partial steps

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Appendix

A Momentum equations

The vector equation can be developed, trying to display terms in $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$.
Advection term becomes :

$$(\nabla \wedge \mathbf{U}) \wedge \mathbf{U} + \frac{1}{2}\nabla \mathbf{U}^2 = \begin{pmatrix} w(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}) - \zeta v \\ w(-\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}) + \zeta u \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{\partial (u^2+v^2)}{\partial x} \\ \frac{\partial (u^2+v^2)}{\partial y} \end{pmatrix} + \begin{pmatrix} w\frac{\partial w}{\partial x} \\ w\frac{\partial w}{\partial y} \end{pmatrix}$$

And :

$$ADV = \begin{pmatrix} -\zeta v \\ \zeta u \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{\partial (u^2+v^2)}{\partial x} \\ \frac{\partial (u^2+v^2)}{\partial y} \end{pmatrix} + \begin{pmatrix} w\frac{\partial u}{\partial z} \\ w\frac{\partial v}{\partial z} \end{pmatrix}$$

Vectorially

$$\left[(\nabla \wedge \mathbf{U}) \wedge \mathbf{U} + \frac{1}{2}\nabla \mathbf{U}^2\right]_h = \zeta \mathbf{k} \wedge \mathbf{U_h} + \frac{1}{2}\nabla_h {\mathbf{U_h}}^2 + w\frac{\partial \mathbf{U_h}}{\partial z} \tag{6}$$

The momentum equation becomes

$$\frac{\partial U_h}{\partial t} = -(f+\zeta)\mathbf{k} \wedge \mathbf{U_h} - \frac{1}{2}\nabla_h {\mathbf{U_h}}^2 - w\frac{\partial \mathbf{U_h}}{\partial z} - \frac{1}{\rho_0}\nabla_h P + \mathbf{D^u}$$

We have $P = P_s + \int_{-z}^{\eta} \rho_0 g dz$, and :

$$\begin{aligned} \frac{1}{\rho_0}\nabla_h P &= \frac{1}{\rho_0}\nabla_h (P_s + \int_{-z}^{0} \rho_0 g dz + \int_{0}^{\eta} \rho_0 g dz) \quad \text{with } P_s = 0 \text{ because free surface} \\ &= \frac{1}{\rho_0}\nabla_h P_h + g\nabla_h \eta \quad \text{with } P_h = \int_{-z}^{0} \rho_0 g dz \end{aligned}$$

The momentum equation can be written :

$$\boxed{\frac{\partial U_h}{\partial t} = -(f+\zeta)\mathbf{k} \wedge \mathbf{U_h} - \frac{1}{2}\nabla_h {\mathbf{U_h}}^2 - \frac{1}{\rho_0}\nabla_h P_h - w\frac{\partial \mathbf{U_h}}{\partial z} - g\nabla_h \eta + \mathbf{D^u}} \tag{7}$$

Projecting on **i** and **j** :

$$\boxed{\begin{cases} \frac{\partial u}{\partial t} = (f+\zeta)v - w\frac{\partial u}{\partial z} - \frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2) - \frac{1}{\rho_0}\frac{\partial P_h}{\partial x} - g\frac{\partial \eta}{\partial x} + D_x^u \\ \frac{\partial v}{\partial t} = -(f+\zeta)u - w\frac{\partial v}{\partial z} - \frac{1}{2}\frac{\partial}{\partial y}(u^2+v^2) - \frac{1}{\rho_0}\frac{\partial P_h}{\partial y} - g\frac{\partial \eta}{\partial y} + D_y^u \end{cases}} \tag{8}$$

B Development of the vorticity equation

Some terms are interesting to develop further :

B.1 $-f\mathbf{k}\wedge\mathbf{U_h}$ term

$$
\begin{aligned}
\mathcal{A} &= \nabla\wedge\frac{1}{H}\int_{-H}^{\eta}-f\mathbf{k}\wedge\mathbf{U_h}dz\\
&= -\left[\frac{\partial}{\partial x}\left[\frac{1}{H}\int_{-H}^{\eta}fudz\right]+\frac{\partial}{\partial y}\left[\frac{1}{H}\int_{-H}^{\eta}fvdz\right]\right]\\
&= -\left[-\frac{1}{H^2}\frac{\partial H}{\partial x}\int_{-H}^{\eta}fudz-\frac{1}{H^2}\frac{\partial H}{\partial y}\int_{-H}^{\eta}fvdz+\frac{1}{H}\left[\frac{\partial}{\partial x}\left(\int_{-H}^{\eta}fudz\right)+\frac{\partial}{\partial y}\left(\int_{-H}^{\eta}fvdz\right)\right]\right]\\
&= \frac{f}{H}\left(U\frac{\partial H}{\partial x}+V\frac{\partial H}{\partial y}\right)-\frac{1}{H}\left[\int_{-H}^{\eta}f\frac{\partial u}{\partial x}dz+\frac{\partial\eta}{\partial x}fu(\eta)+\frac{\partial H}{\partial x}fu(-H)\right]\\
&\quad -\frac{1}{H}\left[\int_{-H}^{\eta}f\frac{\partial v}{\partial y}dz+\int_{-H}^{\eta}\beta vdz+\frac{\partial\eta}{\partial y}fv(\eta)+\frac{\partial H}{\partial y}fv(-H)\right]\\
&= \frac{f}{H}\mathbf{U_h}\cdot\nabla_h(H)-\frac{f}{H}\left[\int_{-H}^{\eta}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dz+\frac{\partial\eta}{\partial x}u(\eta)+\frac{\partial\eta}{\partial y}v(\eta)+\frac{\partial H}{\partial x}u(-H)+\frac{\partial H}{\partial y}v(-H)\right]-\beta V
\end{aligned}
$$

Then, according to (3), we can write :

$$
\begin{aligned}
\int_{-H}^{\eta}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dz &= \int_{-H}^{\eta}-\frac{\partial w}{\partial z}dz\\
&= w(-H)-w(\eta)
\end{aligned}
$$

Using boundary counditions (4 and 5), and neglecting the time derivatives, we have :

$$\mathcal{A}=-\beta V+\frac{f}{H}\mathbf{U_h}\cdot\nabla_h(H)$$

B.2 $-g\nabla_h\eta$ term

$$\frac{1}{H(x,y)}\int_{-H}^{\eta}-g\nabla_h\eta dz=-g\nabla_h\eta \qquad \text{because } \eta \text{ doesn't depend of z}$$

Consequently

$$
\begin{aligned}
\mathcal{F} &= \nabla\wedge(-g\nabla_h\eta)\\
&= (-g)\frac{\partial^2\eta}{\partial x\partial y}-(-g)\frac{\partial^2\eta}{\partial y\partial x}\\
&= 0
\end{aligned}
$$

$$\mathcal{F}=0$$

B.3 Vertical diffusion term

$$\begin{aligned}
\frac{1}{H(x,y)}\int_{-H}^{\eta}\frac{\partial}{\partial z}\left(K_1\frac{\partial \mathbf{U_h}}{\partial z}\right)dz &= \frac{K_1}{H}\left[\frac{\partial \mathbf{U_h}}{\partial z}(\eta)-\frac{\partial \mathbf{U_h}}{\partial z}(H)\right]\\
&= \frac{K_1}{H}\frac{\partial \mathbf{U_h}}{\partial z}(\eta)-\frac{K_1}{H}\frac{\partial \mathbf{U_h}}{\partial z}(H)
\end{aligned}$$

This term is the sum of the wind stress and bottom friction terms. In the model, they are substracted from this term. Consequently, the "vertical diffusion" contribution to the vorticity equation is zero.